5

37

Bibliothèque du Parlement
Classification
Canada

# THE TRISECTION

# OF ANY

# RECTILINEAL ANGLE

Revised August 1911

(1912)

# A Geometrical Problem

# The Trisection
of any
# Rectilineal Angle

By

**Geo. Goodwin**
Contractor,
Ottawa, Canada.

Mr W.R. Billings
Compliments of
Geo Goodwin

Revised August 1911.

OTTAWA
The Mortimer Press
1911

COPYRIGHT, CANADA, 1911
BY
GEO. GOODWIN

# INDEX

PAGE
Addendum (Several methods of Construction)..............17-18
Analysis (Observations preceding) (See Section No. 1)........ 4
Analysis (Section No. 2).............................. 7
Construction (Section No. 1)..........................4-5-6
Diagrams.................................See end of book
Section No. 3, including convergent and divergent values (meaning of defined)...............................8-9-10
Introduction (revised edition)......................... 2-3
Section No. 4, Demonstration Sketch No. 1 and Sketch No. 2, illustrated...............................10-11-12-13
Section No. 5. Lines and Points found without recourse to Hypothesis or Analysis.................13-14-15
Section No. 6. Another method of Construction (*which may be found more expeditious than those preceding*) ..16-17

# Introduction

*In the presentation of this problem there will be some repetition which may be open to objection as unnecessary detail; but it is considered preferable to err in that way than risk the omission of something that ought to be stated.*

*In* 1910 *the author published a solution, the diagram of which was somewhat inconsistent with the principles of the problem. Hence this revised edition.*

Prior to the publication of any Literary or Scientific Work, it is customary to give an outline of the subject in the form of a preface.

In the present instance, the Trisection of any rectilineal angle was first observed by the author several years ago, when reading the following (see page 105 of the Elements of Geometry, in reference to the subdivision of the circumference of a circle, which work is hereinafter more amply identified in the opening page hereof), viz.: “It is obvious that any regular polygon whatever might be inscribed in a circle, provided that its circumference could be divided into any proposed number of equal parts; but such division of the circumference like the trisection of an angle, which indeed depends on it, is a problem which has not yet been effected.”

A possible solution of the problem was suggested by the reading of the following: “If the side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles. (See Euclid 32—1).

It was subsequently discovered that, if a line could be drawn so that the interior and opposite angles would be to each other in the ratio of two to one, the problem could be solved; and that the same result could be attained if a construction could be made by which, as in the following Analysis, the straight line A P would intersect the circumference of the semi-circle in the point V so that V P = A B. During the interval, much time has been periodically devoted by the author to the study of this problem, and with the final result as disclosed in this treatise.

Subsequently to or about the year 1882, Mr. Andrew Doyle published several solutions, and about the year 1895 proved that, if a straight line be drawn from the point A, so that the portion of it situate between a perpendicular such as B F and the circumference of the semi-circle would be equal to the radius, and that to place a line in such position would be equivalent to the solution. Mr. Doyle did not construct to give effect to his theory, and consequently failed to solve the problem.

In justice to Mr. Doyle, it is considered proper for this note of recognition to appear in this publication.

Ottawa, July, 1911.

GEO. GOODWIN.

# Trisection of Any Rectilineal Angle

## SECTION No. 1.

**Clause No. 1.**—As an explanatory note, the qualifying words not greater than a right angle may be used, but do not mean a limitation of the problem, because, if the given angle should be greater than a right angle the angle could be bisected, and the half of the angle, so divided, could be trisected, and the two-thirds of the half would be equal to the one-third of the whole.

In order to facilitate the solution of the several parts of this problem, it has been found convenient to divide it into SECTIONS and CLAUSES, which can be referred to according to the numbers under which they appear.

The geometrical authority, herein quoted, is "The Six Books of "Euclid, by John Playfair, F.R.S., Lond. and Edin. Professor of "Natural Philosophy, formerly of Mathematics, in the University of "Edinburgh. From the last London Edition, Enlarged. Philadel-"phia, 1860."

**Clause No. 2.**—Geometrical writers directly after the Enunciation of a Problem, proceed to show the construction, the demonstration, as to its accuracy, following in regular order. (First Diagram is Construction; main Diagram is Demonstration. See both Diagrams at end of book).

### CONSTRUCTION.

Let the angle A B C be the given angle. Produce the line C B indefinitely (2 Post.) From the point B, in the line B C and with any radius A B describe the semicircle C A D then draw the perpendicular B F (11—1), let A B = B C. Then place the point E, in the line C B produced so that the line B E = 2 A B, then draw the line E A, and from the point H, at the intersection of the line E A with the perpendicular B F, and with a radius equal to B E (or 2 A B) describe the arc intersecting the line C E in the point R. Then draw the straight line A R so far as to intersect the perpendicular B F in the point T and draw the line T′ T L′ through the point T parallel to C E (31—1) intersecting B A in the point T′ and from the point H (in the line A E) draw the line H H′ parallel to C E, (31—1). Then produce the line B A so that A A′ will be equal to T′ H′. Then from the point B with a radius equal to the whole line B A′ describe the semicircle C′ A′ D′. Then from the point H and with a radius equal to B A′ intersecting the semicircle C′ A′ D′ in the point X′ and from that point draw the line X′ B and the angle X′ B C′ will be equal to one-third ($\frac{1}{3}$) of the given angle A B C or $\frac{A\ B\ C}{3}$

## ANALYSIS.

In order to ascertain the conditions that prevail so that the interior and opposite angles may be divided in the proportions of one to two, which if accomplished would be equivalent to the solution of the trisection problem (32—1) the following is quoted from Euclid's Explanations (first book):

16. *Analysis, or the Analytic method, is the art or mode of finding out the truth of a proposition, by first supposing the thing to be done, and then reasoning step by step, till we arrive at some known truth. This is also called the Method of Invention, or Resolution; and is that which is commonly used in Algebra,*

17. *Synthesis, or the Synthetic Method, is the searching out truth, by first laying down simple principles and pursuing the consequences flowing from them till we arrive at the conclusion. This is also called the Method of Composition; and is that which is commonly used in Geometry.*

13. *An Hypothesis is a supposition assumed to be true, in order to argue from or found upon it the reasoning and demonstration of some proposition.*

## BY HYPOTHESIS.

It may be assumed that an angle, as, for example, the angle A B C, is already divided by the line B X so that the angle X B C = $\frac{\text{A B C.}}{3}$ Let A B = B C.

From the point B draw B F perpendicular to C E (or C B) produced (11—1) make B E = 2 C B (or 2 A B). From the point A draw A P parallel to B X intersecting the line C E in the point P (31—1). Then A P B = X B C (28—1) and A B C = A P C + B A P (32—1).

From the point B as a centre and with A B as a radius, describe the semicircle C A D (on C B produced) intersecting A P in the point V. Then B V = A B (being radius of the same semicircle), and the angle B A V = B V A (5—1) and A B C = B A P + A P B (32—1) and B V A = V B P + A P B and by the Hypothesis X B C = A P B and X B C = $\frac{\text{A B C.}}{3}$ The angle A P B = $\frac{\text{B A P}}{2}$ and B A P = B V A (5—1) and B V A = A P B + V B P (32—1), and as A P B = $\frac{\text{B A P}}{2}$ $\frac{\text{B V A}}{2}$ = A P B. From the point V, where the straight line A P intersects the semicircle C A D, draw the line V W parallel to B C (or C E) (31—1). Then B V = B A and B A V = B V A (5—1) and V B P = B V W (27—1) and B V A = 2 A P B. The angle B V A is bisected by the line V W and as C B F is a right angle, the angle V W B is also a right angle because V W is parallel to C E (13—1) or the angle B A P is equal to two of the parts of which the angle A B C contains three.

The alternate angle V B P = B V W and B V A = 2 A P B and A V W = A P B and B V A = A P B + V B P and B V A—A P B = B V A—V B P. Therefore A P B = V B P and V B = V P. It will be seen that the point U is at the intersection of the straight line A U V P (or A P) with the perpendicular B F. The two triangles B V W and U V W have two angles in the one equal to two angles in the other each to each, and the line V W common to both, V W B and U W V are both right angles (13—1), and B V W = U V W (or A V W).

Then there are two triangles having two angles in the one equal to two angles in the other and one side of the one equal to one side of the other (or common to both), then the remaining angle of the one (V U W = V B W) is equal to the remaining angle of the other and the remaining sides of the one are equal to the remaining sides of the other (26—1). Therefore V B = V U and B W = U W.

Furthermore if the line V W be produced it will intersect the semicircle C A D in the point X and if a line be drawn from U to X U X = B X and the triangle U V B is an Isosecles triangle, and the triangle U X B is similar and equal to the triangle U V B and is also an Isosceles triangle and the four lines, viz., B X — X U — U V—V B are equal to each other and also equal to V P and the perpendicular U B is bisected in the point W by the straight line V X (which is parallel to C E) and V X is also bisected in the point W. The line B X is by hypothesis parallel to the straight line A U V P (or A P). The whole figure B X U V B is a Rhombus which is composed of four Isosceles triangles, equal in area and two of each being similar and equal in every respect, viz., B X U and U V B and the other two also, viz., B X V and X U V (5—1). *The two latter overlapping the two former, and vice versa.* The three angles viz., B A V = B V A = B X U are each equal to B A P and by hypothesis B A P = 2 A P B and B A V = 2 A V X (20—3) and A V X = A P B, and A B C = B A P + A P B (32—1) and X B C = A P B and X B C = B X W (27—1) and because U B is bisected in the point W (by the line V X) at right angles to U B and parallel to the straight line C B E the line B W = W U and the line U V = U X and the line X W (in the straight line X W V) is common to the two triangles U W X B W X, both being right angle triangles. The angle B X W = U X W. Then the angle U X B = U X W + B X W and as U X B = B A P and as B A P = 2 A P B the whole angle A B C = X B C + B X W + W X U. The three latter angles have been proved to be equal to each other and the angle B X U is equal to two of the parts of which the angle A B C contains three. Therefore, each of the six angles, viz., X B C, B X W, U X W, U V W, V P B and V B P is equal to $\frac{A\ B\ C}{3}$ and the four lines which form the Rhombus B X U V B, viz.,

B X—X U—U V and V B are equal to each other (def. 27 of 1—1) and are also equal to the portion of A U P that is situate between the points V and P.

It has been shown that V B = V P and U V = V P. The three lines, viz., B V., V P and U V are equal, therefore P V = V U and P V + V U = 2 A B. Note, it will be observed that, if a line be drawn from any other point than from the point P (in the line B E) to the point A, the angle would be less or greater than $\frac{\text{A B C}}{3}$, greater if situate between B and P, and less if situate between P and E.

## SECTION NO. 2.

From the foregoing Analysis the following Theorem is derived, viz.:

If one of the sides of any given rectilineal angle (not greater than a right angle) be produced, indefinitely, and from the point where the two sides forming the given angle intersect each other a line be drawn perpendicular to the side produced, and from that point as a centre a semicircle be described, on the side produced, and if the straight line drawn from the point where the other side of the given angle intersects the circumference (of the semicircle) to a point in the side, so produced, so that the straight line so drawn will be divided by the perpendicular, so that the portion of it that lies, or is situate, between its intersection with the perpendicular and the point where it intersects the side produced will be equal to the diameter (or twice the radius), the angle formed by that line so drawn and the side so produced will be equal to one-third of the given angle and the line so drawn is constant with regard to any angle. The demonstration, in relation thereto, given in the following sections will show that the straight line A P and lines parallel to A P can be so placed independently and without the assumption as set forth by the Hypothesis in Section No. 1.

## SECTION NO. 3.

This section is consistent with Euclid's Explanations already quoted on page 5 hereof.

**Clause No. 1.**—By Hypothesis the straight line A U P forms with C E an angle = $\frac{A\ B\ C}{3}$, and, for the sake of convenience, in reference to the reasoning, the line A P or any line parallel to it or parallel to B X may be referred to and called a trisecting line.

**Clause No. 2.**—The portion of the line A U P that is situate between the perpendicular B F and the point P by Hypothesis is U P = B E (or 2 A B). The angle H B E = U B P (both being right angles) the side U P is greater than the side B P (19—1) and U P = B E. Therefore B E is greater than B P and H E is greater than B E (18—1). H E is included in and is a portion of the line A H E and U P is included in and is a portion of the straight line A U P. The angle A R B is greater than the angle A P B and the angle B A P is greater than the angle B A R and the angle B A E is greater than the angle B A P and A E intersects B F in the point H at a greater distance from B than A P intersects B F in the point U. Therefore B H is greater than B U and B U is greater than B T and the angle A P B is greater than the angle A E B.

**Clause No. 3.**—The following is submitted as explanatory of the undermentioned terms used herein, viz.,—divergent value, convergent value, horizontal convergent value, and vertical value. When certain lines of unequal length are drawn from a point, not situate in the line called the trisecting line, so as to intersect the trisecting line and from that same point if another line be drawn at a different angle so as to intersect the trisecting line in some other point, such lines, while different in length, are said to be of equal value, for example, viz.,—The line E A converges towards the trisecting line and intersects P A in the point A and the line E P converges from E to P, so that those two lines while different in length are said to be of equal value. That is to say, the convergent value of E A is said to be equal to the convergent horizontal value of E P. Then the divergent value of A R is said to be equal to convergent horizontal value of R P.

The meaning of the terms may be further explained, viz.,—The term divergent means a line drawn from any point situate in the trisecting line to any point that is not situate in that line.

The term convergent means a line drawn from any point not situate in the trisecting line to any point that is situate in that line.

**Clause No. 4.**—The term horizontal means the line C E or the distance between any two points situate in C E or any line parallel to it.

The term vertical means the line B F or the distance between any two points situate in B F or any line parallel to it, or any line drawn at right angles to C E or drawn at right angles to any line parallel to C E.

**Clause No. 5.**—The following may be used as illustrative of the foregoing explanations. The line E A converges towards the trisecting line A P and the same line A E diverges from A P and E P converges from E to P, so that the use of the term convergent or divergent is determined by whether the line is supposed to be drawn from E to A or from A to E. The same may be said as to whether the line is supposed to be drawn from R to A or from A to R. The divergent value of A R is equal to the convergent horizontal value of R P or the divergent horizontal value of P R is equal to the convergent value R A. Also the convergent value of H A is equal to the horizontal convergent value of H U′, and the horizontal value of H U′ is equal to the vertical convergent value of H U, and the divergent value of A H is equal to the divergent vertical value of U H.

**Clause No. 6.**—The convergent value between the line A H and the trisecting line A U P measured on the perpendicular B F is H U and when measured on the horizontal L H produced to U′ is equivalent to the convergent horizontal value accrued between the intersection of A P with the horizontal straight line L H U′ and having a horizontal convergent value of H U′, so that the horizontal value of H U′ is equal to the convergent value of H A and the triangle H U U′ is a right angle triangle composed of a portion of the different lines as per sketch No. 1. The lines forming this triangle are viz., H U′, included in and forming a part of the horizontal straight line L H U′ the line H U, included in and forming a part of the perpendicular B F, and the line U′ U, included in and forming a part of the trisecting line A P (or A U′ U P).

Observe H T is greater than H U, therefore the portion of A P situate between H L and T L′ is greater than U′ U. It may be mentioned the line A H converges towards or diverges from the trisecting line A P according as that line is supposed to be drawn from A to H or from H to A.

**Clause No. 7.**—There is, so far as considered, no method (exclusive of the Hypothesis) by which the length of the lines H U′ + R P can be measured or shown horizontally or how either of them can be measured separately. However, it has been shown that if a line be drawn from L to J it would be equal and parallel to H A, and that, as A H = L J, the convergent value of H A is equal to the convergent horizontal value of H U′. See Clause No. 6. Therefore the convergent value of L J is equivalent to the horizontal value of H U′ and by the drawing of the line L J the distance or line equivalent to H U′ would be eliminated, and the line L J has converged from L to J, so that the distance from J to the point where the trisecting line drawn from E would intersect A M in the point J′ would be equivalent to the horizontal value of line R P. The (small) arc at R has been drawn from the point H with a radius equal to the diameter (or 2 A B). The arc B G has been drawn from the point E with a radius equal to the diameter, and R H (which if drawn) would be parallel and equal to E L and A J = R E and R E = L H. The three lines, viz., A J, H L and E R are equal to each other, also because H R is parallel to E L (and between the parallel lines H L and C E), the line R E = H L (33—1). Therefore the three lines, viz., R E = H L and = A J are equal to each other (see Axiom No. 1, Book 1) "things which are equal to the same thing are equal to one another," A R is parallel and equal to E J (33—1).

It has been shown that L J, if drawn, would converge towards the line drawn from E parallel to A P so as to eliminate the equivalent of H U′, so that the convergent value of H A is equal to the horizontal value of H U′. The horizontal difference measured along L H from L towards H, or, in other words, the difference between the line E L and the line drawn from E parallel to P A and intersecting the line L H, would be equal to U′ H + R P but R P = J J′. The line drawn from E parallel to P A will intersect the line L H at a horizontal distance from the point L = U′ H + J J′. This is evident because if a line be drawn from P parallel to R H it will show the horizontal equivalent of R P on H L which together with H U′ will = H U′ + R P (or H U′ + J J′).

## SECTION NO. 4.

### Demonstration and Sketches Illustrated.

**Clause No. 1.**—It is now to be determined to what distance the straight line B A shall be produced so that it will intersect the straight line drawn from the point H parallel to the line P A. Let A B be produced to the point A′ according to Construction (see Sec. No. 1). A A′ = T′ H′ and B A′ = B A + T′ H′. It may be mentioned B A is included in and is a portion of the straight line B A′ and coincides therewith.

The lines H L and T L′, also the lines A M and A′ M′ are by construction parallel to B E and are therefore parallel to each other (30—1). The lines A M and A′ M′ are (by construction) the same distance from each other as are the lines H L and T L′.

**Clause No. 2.**—The, so called, intervening line marked U U″ in Sketch No. 1 is situate between the lines H L and T L′, and the other, so called, intervening line marked O N in Sketch No. 2 is situate between the lines A M and A′ M′. Both of these intervening lines are parallel to the lines between which they are situate, and by Hypothesis the said intervening line U U″ is situate at a greater distance from H L than it is from T L′ and the intervening line O N is situate at a greater distance from A M than it is from A′ M′ and U U″ is situate at the same distance from T L′ that O N is situate from the line A′ M′. The intervening line U U″ so nearly coincides with T L′ and the intervening line O N so nearly coincides with the line A′ M′ that it would be difficult (particularly if a smaller angle had been given) to observe any difference in position between these intervening lines and the lines with which they so nearly coincide. The two sketches marked Sketch No. 1 and Sketch No. 2 on the main diagram are necessarily drawn in an enlarged form and are for the purposes of illustration to enable the reader to follow without difficulty. The lines in these sketches are lettered in the same manner as the lines on the main diagram which they represent.

**Clause No. 3.**—The lines A M and A′ M′ are the same distance from each other as are the lines H L and T L′ (see Clause No. 1). The distance from A M to H L is common, therefore the whole distance between the parallel lines A M and T L′ is equal to the whole distance between the parallel lines A′ M′ and H L.

Therefore that portion of the straight line A P situate between the point A and its intersection with the line T L′ is equal to that portion of the line parallel to A P drawn from the point H to its intersection with the line A′ M′.

Therefore "the straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also equal and parallel" (33—1). (Observe Sketch No. 2) A A′ = H H″.

**Clause No. 4.**—Suppose a line be drawn from the point H parallel to the trisecting line A P (or A U P) and produced so as to intersect the line B A produced it will intersect the perpendicular A A″ raised from the point A at the same distance from A that H is from U. (Observe SKETCH No. 2) so that the perpendicular H U = A O′ and the line A U = O′ H; and the four-sided figure A U H O′ A is a parallelogram with the opposite sides and angles equal to one another (34—1). The angle A U H = H O′ A. Through the point H draw the line K K′ parallel to B A (31—1) intersecting the line T L′, in the point indicated (see SKETCH No. 1) by H″. Through the point H draw the line H H′ parallel to B E intersecting B A in the point H′ and through the point T draw T T′ parallel to B E, intersecting B A in the point T′ (31—1). Then K K′ is made parallel to B A and H H′ is parallel to T T′; and H′ T′ is parallel and equal to the line H H″ which is included in K K′ (see SKETCH No. 1) and by construction A A′ = T′ H′ and A A′ = H H″. Therefore the four-sided figure A H″ H A′ A is a parallelogram and A′ H is parallel and equal to A H″. A H″ = A′ H and A A′ = H H″ and the straight line A H″ is included in and is a part of the whole straight line A P (or A U H″ P) and the line A′ H is included in and is a part of the whole straight line A′ P′ (or A′ H P′) which (according to Section No. 2) is equal to A′ H + 2 B A′ and likewise if from the point H and with a radius equal to 2 B A′ an arc be described it will intersect the line B E so that the line drawn from the point H to that point of intersection will form an angle equal to $\frac{A\ B\ C.}{3}$ The vertical distance between the intersection of A E and A P with the perpendicular B F is the vertical value H U, and the whole of the divergence between lines A E and A R at their intersection with the perpendicular B F is the vertical divergent value of H U + U T and H U + U T = H T.

The portion of A P situate between the lines H L and U U″ is less than that situate between the lines H L and T L′. As will be seen, the difference between H T and H U is U T which is taken from the lower part of H T and is indicated by the space between the intervening line U U″ and the line T L′. In the case of the perpendicular A A″ from A M to A′ M′, the difference is taken from the upper part which is the space between the intervening line O N and A′ M′ which is also equal to the space between the intervening line U U″ and T L′. which are each equal to the divergence of the line A R from A P that accrues between the point A and the intersection of A R with the perpendicular B F. (See SKETCH No. 1 and SKETCH No. 2).

The right angle triangle (marked on SKETCH No. 1) described in Section No. 3, Clause No. 6, as will be seen, is situate between the lines H L and U U″, and another right angle triangle, but in an inverted form, it will be observed (by reference to SKETCH No. 2) is situate between the lines A M and O N.

The portion of the straight line H A′ situate between its intersection with the line A M and its intersection with the perpendicular A A″ is equal to 2 A A′ together with the remaining portion of the straight line situate between its intersection with the perpendicular A A″ and the point A′( which is indicated by O′ A′), and the portion of the straight line A P (or A U H″ P) situate between the point U′ and the point U is equal to 2 H H″ together with the portion of the line A P situate between the point U (in the perpendicular H T) and the point H″, that is to say O′ A′ = U H″. A reference to SKETCH No. 1 and SKETCH No. 2 will show this theory is established. (See Theorem, Section No. 2).

## SECTION No. 5.

**Clause No. 1.**—In pursuance of the analysis in Section No. 1 regarding the straight line A U P forming with the line C E an angle equal to (⅓) one-third of the given angle, the point U, as will be observed, is found by the hypothesis and therefore does not comply with the conditions of the problem.

**Clause No. 2.**—According to the construction already enunciated the points H and A′ are found independently and without having recourse to hypothesis. As will be observed the arcs described from the point H with a radius equal to the line B A′ intersect the semicircle C′ A′ D′ in the points X′ and V′. The straight line drawn through the points X′ and V′ is parallel to the line B E and intersects the perpendicular B F in the point W′, which bisects the line B H, so that the straight lines X′ W′ V′ and H B bisect each other at right angles in the point W′ (Prob. 9 & 10—1 together with Theor. E 37—3 is authority for that theory.)

**Clause No. 3.**—The line H X′ being made equal to B A′. The triangle B H X′ is an Isosceles triangle and H X′ = H V′ and B X′ = B V′ (5—1).

The perpendicular H B (in the line B F) is common to both of the Isosceles triangles, viz., B H X′ and B H V′. The four lines, viz., B X′ —H X′—H V′ and B V′ are equal to each other so that the four-sided figure B X′ H V′ B is a Rhombus (def. 27 1—1) and is composed of four Isosceles triangles equal in area and two of each being equal and similar in every respect, viz., B X′ H and H V′ B and the other two also, viz., B X′ V′ and X′ H V′ (5—1). *The two latter overlapping the two former, and vice versa.* Then the line X′ V′ in the other two Isosceles triangles is also common to both, viz., B X′ V′ and X′ H V′, which triangles are also equal and similar in every respect. By construction H V′ = B A′ and 2 B A′ = H P′ because the perpendicular B H = 2 W′ H. The angle H W′ V′ is a right angle, the angle H B P′ is also a right angle and the triangle H W′ V′ is similar to the triangle H B P′, and as H B = 2 H W′ and X′ W′ = W′ V′ then B P′ = X′ V′ and the triangle B V′ P′ is an Isosceles triangle, B V′ = P′ V′.

Furthermore, it is clear that the sum of all the interior angles of a Rhombus are together equal to four right angles. Therefore, two of the angles adjacent to any one of the sides, as, for example, the line H X′, the angles H X′ B and X′ H V′ are together equal to two right angles. The angle V′ H W′ + H V′ W′ = one right angle (or V′ W′ H = W′ H L) and L H W′—V′ H L = V′ H W′ and W′ H H′ is a right angle, so that the straight line V′ A′ intersects the perpendicular B F so that the angle V′ H W′ (or V′ H B) is as much less than a right angle as the angle A′ H W′ (or A′ H B) is greater than a right angle. Therefore the angle V′ H B + B H A′ is equal to two right angles.

**Clause No. 4.**—It is required to show that the points A′, H and V′ are in a straight line. Because the triangle V′ B P′ is an Isosceles triangle (see Clause 3) similar and equal in every respect to the two last just described. B X′ = B V′ and H V′ = V′ P′ and X′ V′ = B P′, B A′ = B V′ (5—1).

The triangle B A′ V′ is also an Isosceles triangle. The angle A′ V′ B = V′ B P′ + B P′ V′ (32—1) and the angle H V′ X′ = H P′ B and the angle X′ H V′ = X′ H B + B H V′ and the angle X′ H V′ + H X′ V′ + H V′ X′ is equal to two right angles "and the three interior angles of every triangle are equal to two right angles" (32—1), also because H V′ B = 2 H V′ X′ (or X′ V′ H + V′ X′ H), the two angles, viz., X′ H V′ + B A′ V′ are equal to two right angles. The angle X′ H V′ + A′ V′ B (or B V′ A′) are also equal to two right angles. The angle V′ H W′ + B H A′ are also equal to two right angles.

As shown in clause No. 3 hereof the triangle H W′ V′ is similar to the triangle H B P′ and H B = 2 H W′ and H P′ = H V′ + V′ P′ (or 2 H V′) (2—6.)

Again suppose, if possible, that the lines V′ H and H A′ are not in a straight line and that the line B A would have to be produced to a greater distance than the point A′ or to a less distance than the point A′. Then in the former case the line B V′ would be less than B A′ produced, and in the latter case the line B V′ would be greater than B A produced to any point situate between the points A and A′, but according to Theorem, Section No. 2, B A′ = B V′ and if V′ H be produced to any point at a greater distance than B A′ or to any point at a less distance than B A′, the straight line P′ V′ H produced cannot intersect B A produced at any other point than the point A′.

Therefore, the points A′, H and V′ are in a straight line because "if at a point in a straight line two other straight lines upon the opposite sides of it make the adjacent angles together equal to two right angles, the straight lines are in one and the same straight line" (14—1). Therefore the lines V′ H and H A′ are in the same straight line V′ A′.

**Clause No. 5.**—The line X′ V′ is situate at an equal distance between the points H and B. The straight line A′ V′ produced to intersect B E in the point P′ (as in Section No. 1 in which the Rhombus is similar to the Rhombus in this Section No. 5). The line H V′ = V′ P′ and B A′ = H V′ and the whole line H P′ = 2 H V′ (or H V′ = V′ P′). The angle B A′ P′ = B V′ A′ (5—1) and the three angles B A′ V′ and B V′ A′ and B X′ H are each equal to B A′ P′ (or B A′ V′).

According to the analysis of the problem the angle B A′ P′ = 2 A ′P′ B and A′ B C′ = B A′ P′ + A′ P′ B (32—1).

The angle A′ B C′ is equal to the given angle A B C. The difference is that the straight lines B A and B C and B X if produced would coincide with the points A′, C′ and X′ (so that A A′ = C C′ = X X′) and the perpendicular B U if produced to the point H. Therefore, the difference between the four sides of the Rhombus in pursuance of the hypothesis in Section No. 1 and the Rhombus in this Section No. 5, is equal to A A′ (or C C′ or X X′) so that the Rhombus described in each of the sections is similar to each other and therefore equiangular and the vertical or perpendicular difference is H U (in the perpendicular B F) and the vertical difference or distance between the lines X V and X′ V′ is W W′ or $\frac{H\ U}{2}$ and the angle B X′ H is equal to two of the parts of which the angle A′ B C′ contains three. Therefore the six angles, viz., X′ B C′ — B X′ W′ — H X′ W′ — H V′ W′ — V′ P′ B — V′ B P′ are equal to each other and also equal to the angle $\frac{A'\,B\,C'}{3}$ The four lines that form the Rhombus B X′ H V′ B, viz., B X′ — X′ H —H V′—V′ B are equal to each other (def. 27 of 1—1) and also equal to the portion of A′ H P′ situate between the points V′ and P′. It has been shown that the portion of the straight line A P situate between U and P is equal to 2 B A and P V + U V = 2 A B and the portion of A′ P′ situate between H and P′ = 2 B A′, therefore the difference between U P and H P′ = 2 A A′

The angle B A′ V′ = B V′ A′ (5—1). The angle H X′ B = H V′ B (5—1) and the whole A′ B C′ = X′ B C′ + X′ B A′ and the angle X′ B A′ is bisected by the line B B′ and the angle X′ B B′ = A′ V′ X′ (20—3) or X′ B A′ = 2 A′ V′ X′ (20—3) A′ V′ X′ = H X′ V′ (5—1) and A′ V′ B is bisected by the straight line X′ V′ (or X′ W′ V′) so that the angle A′ V′ X′ = B V′ X′ and the angle B X′ V′ = B V′ X′ (5—1)

The four angles, viz., A′ V′ X′ = B V′ X′ = H X′ V′ = B X′ V′ are equal to each other. And X′ B C′ = B X′ V′ (27—1) and A′ B B′ = A′ V′ X′ and X′ B B′ = A′ V′ X′ and X′ B C′ = B X′ W′ (27—1) and B H is bisected in the point W′ (by the line V′ X′) at right angles to B H (and parallel to B E). B W′ = W′ H and H V′ = H X′ and X′ W′ (in the straight line X′ W′ V′) is common to the two triangles H X′ W′ and B X′ W′ and the angle H W′ X′ = B W′ X′ both being right angles.

The angle B X′ W′ = H X′ W′, then the angle H X′ B = H X′ W′ + B X′ W′, so that H X′ B is bisected by the line X′ W′ and H X′ B = B A′ P′ (or B A′ V′) and as B A′ P′ = 2 A′ P′ B, the whole A′ B C′ = X′ B C′ + B X′ W′ + W′ X′ H. The three latter angles have been proved equal to each other.

X′ B A′ = 2 A′ V′ X′ (20—3) and X′ B A′ = A′ B B′ + B′ B X′ and 2 A′ V′ X′ = A′ B B′ + B′ B X′. Therefore the angle A′ B C′ has been divided into three equal parts by the two lines B X′ and B B′.

## SECTION NO. 6.

**Clause No. 1.**—This Section is another method adopted to determine the point where the trisecting line drawn from the point E intersects L H.

Produce the line E L to intersect A M in the point Y and from the point J (in the line A M) draw the line J Z parallel to E L intersecting the line E A in the point E′. The straight line J E′ Z will intersect the line L H in the point S so that the line drawn from E to the point S will be parallel to A P, and form an angle with the line C E that will be equal to one-third of the given angle A B C.

It has been shown (see last part of clause 7, Section No. 3) that the horizontal divergent value of E L from the trisecting line drawn from E to its intersection with H L, measured from L along the line L H, is equivalent to H U′ + J′ J. Then E L being produced to intersect A M in the point Y, and the line J E′ produced to intersect C E in the point Z and the straight line J E′ Z being drawn parallel to the straight line E L Y, E Z = J Y.

From the point Y (in the line Y E) make Y Y′ = E L. Then E Y′ = L Y, H A = E E′, H A = L J (see Section No. 3, Clause No. 4). Then E E′ = L J, J E′ = E L, E L = Y Y′. The divergence of E L from the trisecting line drawn from the point E to its intersection with the line H L has been shown to be equal to the horizontal value of U′ H + J′ J. Then E L = Y Y′ and E L + Y Y′ = 2 E L and 2 E L is equal to the horizontal value of 2 U′ H + 2 J′ J.

It will be observed that E L + L Y is equal to the whole of the line E Y, but 2 E L is greater than the line E Y, 2 E L = E Y + L Y′ and 2 E L — L Y′ is equal to the horizontal value of U′ H + 2 J′ J, which shows L Y′ is equivalent to the horizontal value of U′ H, and L Y′ is equal to that portion of E L situate between the points Y′ and L, and the line drawn from E′ to the intersection of the trisecting line with H L will be equal and parallel to the line Y′ L or Y′ L = E′ S.

As already shown, E Y = J Z and J Z = Z E′ + E′ S + S J. Then Z E′ = E Y′, Y′ L = E′ S and S J = L Y. Therefore the divergent value of E L is equal to or equivalent to the horizontal convergent value of Z S and E′ S = Y′ L. Then the horizontal line L S is equal to the horizontal line E Z (or L S = E Z) and L S = U′ H + J′ J and the divergent value of Y′ L is equal or equivalent to the horizontal value of U′ H and Y′ L = E′ S.

Furthermore, if a line parallel to E L be drawn from the point J′ (in the line A M) instead of from the point J, it will intersect the line H L at a horizontal distance measured from the point L that will be equal to U′ H + 2 J′ J instead of U′ H + J′ J. Therefore it is clear that the line E L produced to Y (in the line A M) diverges from the trisecting line drawn from E to J′, and Y J′ = U′ H + 2 J′ J. This reasoning establishes that the divergence of L Y is equal or equivalent to the horizontal J′ J.

**Clause No. 2.**—The principle involved in this section, as already shown, is that the divergence of E L from the trisecting line drawn from the point E to its intersection with the line L H is the horizontal equivalent of U′ H + J′ J (see Clause No. 1), and that the line E L is divided in the point Y′ so that L Y′ is the horizontal equivalent of U′ H, and E Y′ is the horizontal equivalent of J′ J.

Therefore, if a line equal to L Y be measured from the point L along the line E L and from that point of section, viz., Y″, a line be drawn parallel to L J, it will intersect L H in the same point, viz., S, that the trisecting line, drawn from the point E, intersects L H, and the line drawn from the point J parallel to E L will also intersect L H in the same point S that the trisecting line drawn from the point E intersects the line L H. As already stated, the straight line J Z has been drawn from the point J parallel to the line E L, and it is concluded that a line drawn from the intersection of that line with L H to the point E will form an angle with the line C E that will be equal to one-third of the given angle A B C.

---

## ADDENDUM.

There are other constructions by which the Trisection Problem can be solved and explained by reference to the main diagram, which are consistent with the principles involved in the solution of the problem as already presented, viz.:

1. From the point H (at the intersection of the line A E with the perpendicular B F) and with a radius equal to 2 A B + 2 A A′ describe an arc intersecting the line C E, and the line drawn from that point of intersection to the point H will form an angle with the line C E that will be equal to one-third of the given angle A B C.

2. The line R H, if drawn, would be equal and parallel to the line E L. From the point R in the line R H, assumed to be drawn, make R R′ = E Y′ (see Section No. 6) and from the point A draw A R′ produced to intersect the line C E and that line will form an

angle with C E that will be equal to one-third of the given angle A B C, that is to say it will intersect C E in the point P.

3. From the point where the arc B G intersects the line T L′ erect a perpendicular (11—1) to intersect the line L H.

Then take the line situate between the point H and the point where the line A R intersects the line L H produced. The sum of these two lines, as described, the former, from L to the intersection of the perpendicular (erected from the intersection of the arc B G with T L′) being greater than R P and the latter being greater than H U′, their sum is greater than H U′ + R P, and R P + U′ H measured from L along the line L H, is equal to the horizontal distance from L that the trisecting line, if drawn from the point E, would intersect the line L H. Then the line drawn from the point J to that point where the trisecting line would intersect L H is equal or equivalent to the divergent or convergent horizontal value of J′ J (or R P).

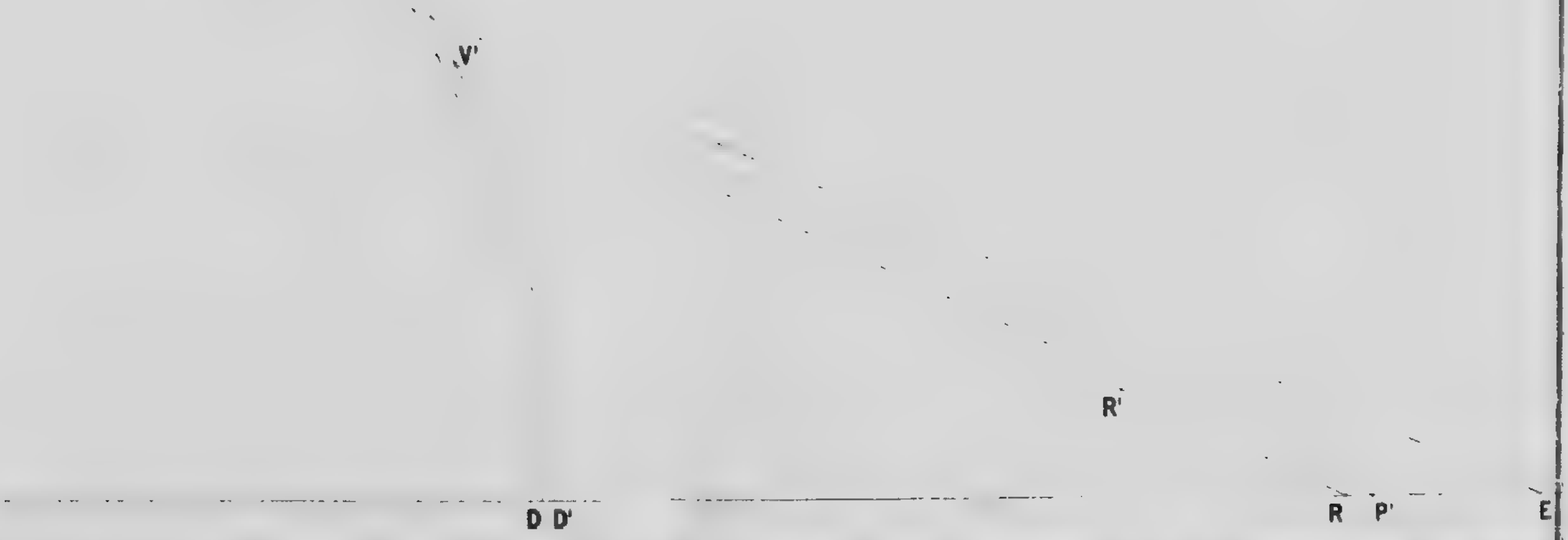
Therefore, if a line equal to the sum of these two lines, described, be measured from the point L along the line L H, it will intersect L H at a greater distance from L than the trisecting line, drawn from E, intersects L H. Then if a line be drawn from J to this latter greater distance as described, in L H from L, that line will have a greater convergent or divergent horizontal value than J′ J (or R P). Then draw from the point E a line parallel and equal to the one just described as having a greater divergent or convergent horizontal value than R P, and from the extremity of that line draw a line parallel to E L intersecting the line H L produced, and from that point of intersection with H L (produced) take a distance equal to the sum of the two said lines (2-1) (which is greater than R P + H U′), and that will give the point where the trisecting line drawn from the point E intersects the line L H (or H L).

4. The angle A′ H H′ is equal to one third of the given angle A B C.

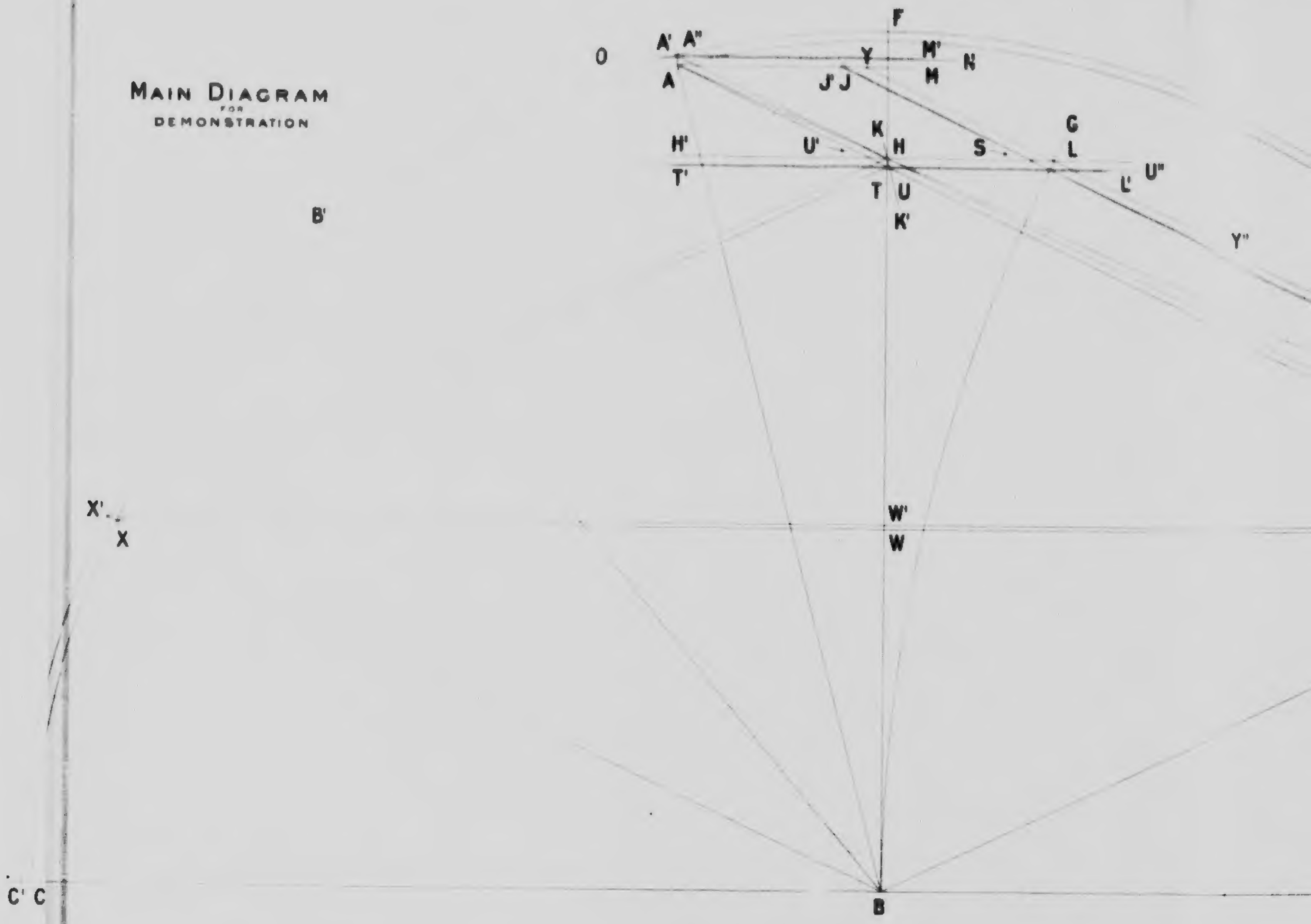
5. The line K K′ being drawn through the point H, parallel to the line B A intersecting the line T′ L′ in the point H″, the angle A H″ T′ is equal to the one third of the given angle A B C. (See sketches and main Diagram).

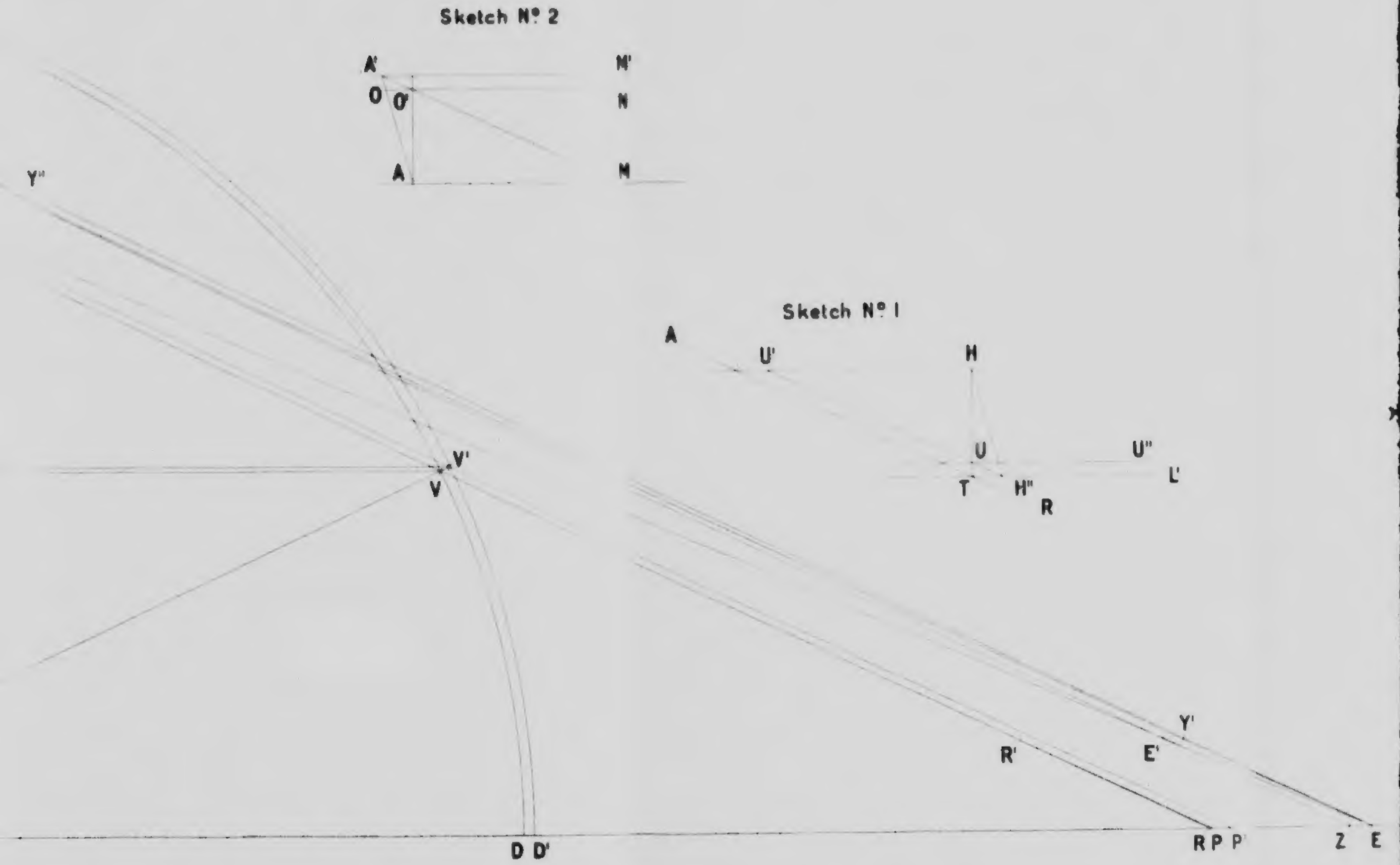
THE END.

## FIRST DIAGRAM FOR CONSTRUCTION


V'
R'
D D'
R
P'
E

# MAIN DIAGRAM
FOR
DEMONSTRATION




Sketch Nº 2
A'
M'
O
O'
N
A
M
Y''
V'
V
Sketch Nº 1
A
U'
H
U
U''
T
H''
R
L'
Y'
E'
R'
D
D'
R
P
P'
Z
E